

Quantum Entanglement Percolation

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Quantum communication demands efficient distribution of quantum entanglement across a network of connected partners. The search for efficient strategies for the entanglement distribution may be based on percolation theory, which describes evolution of network connectivity with respect to some network parameters. In this framework, the probability to establish perfect entanglement between two remote partners decays exponentially with the distance between them before the percolation transition point, which unambiguously defines percolation properties of any classical network or lattice. Here we introduce quantum networks created with local operations and classical communication, which exhibit non-classical percolation transition points leading to the striking communication advantages over those offered by the corresponding classical networks. We show, in particular, how to establish perfect entanglement between any two nodes in the simplest possible network – the 1D chain – using imperfect entangled pairs of qubits.

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I. INTRODUCTION

Rapid development of quantum technologies implies that large networks operating with quantum information are to be created in the nearest future [1]. Such quantum networks are superior to their classical ancestors in security [2] and efficiency of communication [3] and exhibit new structures [4] and behavior [5], which may be exploited in quantum computation [6] and metrology [7]. In general, a network is represented by a graph with nodes connected by links. In quantum networks, the links are often associated with quantum channels, through which the nodes exchange photons [8]. The photons may be locally measured with subsequent classical communication of the results of the measurements between the nodes.

For efficient communication in a quantum network with certain configuration of nodes and links, perfect long-distance entanglement between arbitrary nodes is to be established. As originally suggested by Acín, Cirac and Lewenstein [9], methods of classical percolation theory [10] can be employed to find the best strategy for entanglement distribution in a network of particular configuration leading to the notion of classical entanglement percolation. Within this notion the communication capacity of a network is described by its percolation transition point, before which the probability to establish a path of perfect entanglement links between two remote nodes is exponentially small, while this probability has a finite asymptotic limit right after it. Since each network configuration relates to the transition point, classical entanglement percolation imposes the fundamental limit on the communication capacity of the network. This limit can be overcome by global change of the network configuration with local operations and classical communication

(LOCC) [9, 11–15]. In particular, by implementing LOCC on a fraction of selected nodes of the network of initial configuration, such as a honeycomb lattice, one may redistribute the entanglement between the nodes of a network of final configuration, e.g. triangular lattice [9]. This quantum network reconfiguration reduces the percolation transition point making quantum communication possible at large scale even with initially insufficient amount of distributed entanglement. The price for this advantage is that the selected nodes become disconnected from the network of final configuration. In other words, the selected nodes of the initial network must sacrifice their connectivity for communication benefit of the final network. From communication viewpoint, this may not be always appreciated at large scale and especially by the selected nodes. In addition, such a strategy cannot be implemented on a network of arbitrary initial configuration. For example, given a triangular lattice or a 1D chain there is no way to execute the above procedure.

Here we suggest a different approach to overcome the limitations of the classical entanglement percolation. We employ LOCC to create complex quantum networks with new percolation properties on networks of simple initial configuration. Our approach is based on multiple applications of LOCC and is free of sacrificing nodes of the initial network. We show, in particular, that the simplest 1D quantum network with the percolation transition point at unity, can be modified to the complex so-called hierarchical network [16, 17] with the transition point at $1/2$ using at most polynomial number of LOCC. In the following section we show first how to transform the 1D chain into the hierarchical network using LOCC. Then, we give an account on physical resources required for the transformation in Section III and conclude in Section IV.

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II. ENTANGLEMENT PERCOLATION IN 1D CHAIN

Let us consider a 1D chain, where nodes are placed on a line at fixed distances from each other and are connected by channels. Two neighboring nodes may exchange photons through the channels and thus share pure entangled states of qubits, which may be written in the computational basis [18] as

$$|\varphi\rangle = \sqrt{\lambda_1} |00\rangle + \sqrt{\lambda_2} |11\rangle, \quad (1)$$

where λ_1 and λ_2 are the Schmidt coefficients conditioned by $\lambda_1 \geq \lambda_2$ and $\lambda_1 + \lambda_2 = 1$. A perfect entangled pair with $\lambda_1 = \lambda_2 = 1/2$ can be converted from the above state with the singlet conversion probability $p = 2\lambda_2$ by measurement of one of the qubits from the pair with operators [19]

$$M_1 = \begin{pmatrix} \sqrt{\frac{\lambda_2}{\lambda_1}} & 0 \\ 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} \sqrt{1 - \frac{\lambda_2}{\lambda_1}} & 0 \\ 0 & 0 \end{pmatrix}. \quad (2)$$

In quantum communication the singlet conversion probability plays the exact role of link occupation probability in percolation theory. Indeed, if the imperfect entangled pairs are distributed between the nodes of a quantum network, each of the pairs is converted to the perfect entanglement link with probability p or vanish with probability $1 - p$. By analogy with the percolation theory, we say that perfect entanglement is established between two remote nodes if there is a path of connected perfect entanglement links between the nodes. While the singlet conversion probability is a natural choice to study the entanglement percolation, we shall also use another measure of entanglement to characterize the process of entanglement distribution – the concurrence [20]. The concurrence for the entangled qubit pair (1) is given by $\mathcal{C} = 2\sqrt{\lambda_1\lambda_2}$. Perfect entanglement with concurrence $\mathcal{C} = 1$ is established between two distant nodes if only there is a path of connected entanglement links of concurrence $\mathcal{C} = 1$ each.

The main ingredient for the advanced quantum information processing is the entanglement swapping [21]. If three nodes $a - b - c$ are chained together and the node b shares two qubit pairs in state (1) with the neighbors, the optimal entanglement swapping [12] consists of the measurement of the two qubits from the two different entangled pairs at node b in the Bell basis

$$\begin{aligned} |\Psi^\pm\rangle &= \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle), \\ |\Phi^\pm\rangle &= \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle), \end{aligned} \quad (3)$$

with subsequent classical communication of the results of the measurement to nodes a and c (see Fig. 1). The two local entanglement links transform into a single non-local

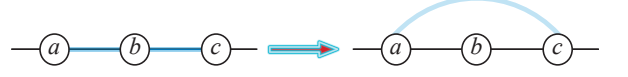


FIG. 1: Entanglement swapping at node b leads to the creation of a non-local entanglement link connecting two physically disconnected nodes.

entanglement link with the final state given by

$$|\psi^\pm\rangle = \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}} |00\rangle \pm \frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} |11\rangle, \quad (4)$$

$$|\phi^\pm\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle), \quad (5)$$

with corresponding probabilities $(\lambda_1^2 + \lambda_2^2)/2$ and $\lambda_1\lambda_2$. The entanglement swapping doesn't change the average singlet conversion probability $p_{\text{swap}} = p$ [22] and reduce concurrence $\mathcal{C}_{\text{swap}} = 2\lambda_1\lambda_2 \equiv \alpha\mathcal{C}$ [12], where $\alpha = \sqrt{\lambda_1\lambda_2}$. The possibility to use LOCC to create a non-local entanglement link, which is beyond the initial network configuration, allows us to create a complex quantum network with highly unexpected large-scale behavior as we show next.

Let us consider a 1D chain consisting of N nodes. Let each pair of neighboring nodes initially share K qubit pairs in the state (1). Let us take $K - 1$ qubit pairs and perform entanglement swapping at each second node. After that we take $K - 2$ qubit pairs and apply the entanglement swapping at each third node. Repeating the above procedure with the entanglement swapping at $2^k + 1$ nodes for $k = 0 \dots K - 2$, we construct $K - 1$ levels of hierarchy of non-local entanglement links over the initial 1D chain formed with local links, as exemplified in Fig. 2. This hierarchical network was introduced by Boettcher, Goncalves and Guclu [16] to study the behavior of the so-called small-world systems [23]. But, because the non-local entanglement links are created with LOCC only and do not require any non-local interactions between the nodes, they cannot be attributed to the small-world links in a common classical sense. Moreover, the non-local links don't correspond to physical communication channels: the physical connectivity of the chain is still one-dimensional. Thus, created hierarchical net-

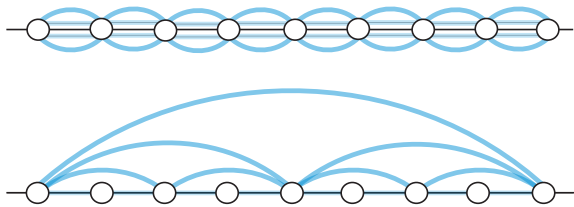


FIG. 2: The 1D chain of 9 nodes with 4 entangled pairs between neighboring nodes (top) is modified with the entanglement swapping at each node - ignoring the borders - to the quantum network with 3 levels of hierarchy (bottom).

work of entanglement links has no classical analog and may be called a genuine quantum network constructed on a 1D chain, while the entanglement percolation in such a network – the quantum entanglement percolation. The suggested approach demands creating non-local entanglement links beyond the given network configuration and cannot be implemented classically.

Now, we show that quantum entanglement percolation in the hierarchical quantum network is of exponential benefit over the entanglement percolation in 1D chain. In the 1D chain with K entangled pairs between two neighboring nodes, perfect entanglement between two neighboring nodes is established with probability $p' = p + p(1 - p)^K$. The condition for connecting two infinitely distant nodes with a perfect entanglement link reads as

$$\lim_{N \rightarrow \infty} (p')^N = 1, \quad (6)$$

where N is the number of nodes between the nodes to be connected. The condition holds true only for $p' = p = 1$ giving us the well-known percolation transition point in the classical 1D chain $p_t = 1$, i.e. in 1D configuration the perfect entanglement can be created between two infinitely distant nodes if only the perfect entanglement is established between each pair of neighboring nodes.

For the quantum hierarchical network, the probability to connect two neighboring nodes is $P_0 = p$, while for three nodes $P_1 = p + (1 - p)P_0^2$, where we used the fact that the entanglement swapping doesn't change the singlet conversion probability. Taking into account self-similarity of the chain, we arrive to the recursive formula to find the probability of connecting the border nodes with a path of perfect entangled pairs [17]

$$P_{k+1} = p + (1 - p)P_k^2, \quad (7)$$

for $k = 0 \dots K - 2$. In the infinite chain $N \rightarrow \infty$, thus $K \rightarrow \infty$ and $P_\infty = P_{K-1} \approx P_{K-2}$. The quadratic equation has two solutions interconnected at the non-classical percolation transition point $p_t = 1/2$. It is important to stress that, in contrast to standard percolation [10], before the percolation transition point $p < 1/2$ there is a finite probability of connecting the border nodes $P_\infty = p/(1 - p)$. However, further analysis [24] shows, that the probability to connect an arbitrary two nodes of the network with a path of perfect entanglement links is still exponentially small due to exponentially small size of the giant component with respect to the network size. For $p > 1/2$ a path between arbitrary two nodes in the network exists with a finite probability [24]. Thus the initially distributed entangled states (1) in the hierarchical network with the singlet conversion probability of $p \geq 1/2$ are sufficient to establish a path of perfect entanglement links between arbitrary two nodes irrespective of the chain length. Taking into account that in the 1D chain the perfect entanglement path exists only if $p = 1$ (and otherwise $P_\infty = 0$), for any $p > 1/2$ quantum entanglement percolation is superior exponentially to the classical entanglement percolation as was announced.

In Eq. (7) we implicitly assumed that the average singlet conversion probability doesn't change after multiple entanglement swapping operations. But, this is not the case, because after a single entanglement swapping with the Bell measurements (3), the pure state (3) transforms into a mixture of states (4) and (5). Further application of the entanglement swapping on the mixed state leads to the decrease of the average singlet conversion probability. Since the entanglement swapping is a LOCC and the entanglement doesn't increase under LOCC, there must be an optimal measurement that preserves the entanglement under multiple application of the entanglement swapping. But, as the optimal measurement is unknown yet [12], in the next section we shall estimate the amount of initial entangled pairs (1) and number of simple entanglement swapping operations with the Bell measurements (3) to create the hierarchical network on the 1D chain.

III. PROPERTIES OF THE ENTANGLEMENT PERCOLATION

Before we proceed with the percolation properties of the quantum network, we would like to stress that each of the $K - 1$ levels of hierarchy connects $2^{K-1} + 1$ nodes into loops with one entangled link between the border nodes. Thus to construct a network of maximal hierarchy on N nodes (i.e. to connect the border nodes with a single entanglement link) one needs to distribute initially at least $1 + \log_2(N - 1)$ imperfect entangled pairs between each pair of neighboring nodes in the initial 1D chain. The total number of the imperfect entangled states in the hierarchical network of N nodes thus scales as $N(1 + \log_2(N - 1)) \ll N^2$, which is practically feasible.

Studying the quantum entanglement percolation with concurrence gives us further insight into the process of entanglement distribution in the quantum network. Because single entanglement swapping reduces the concurrence of the initial states $\alpha < 1$ times, the recursion relation (7) is modified as

$$P_{k+1} = \alpha^{k+1}\mathcal{C} + (1 - \alpha^{k+1}\mathcal{C})P_k^2, \quad (8)$$

where $P_0 = \mathcal{C} = 2\sqrt{\lambda_1\lambda_2}$. Because $\alpha^{k+1}\mathcal{C} \rightarrow 0$ as $k \rightarrow \infty$, the percolation properties of the hierarchical network reduce to the percolation properties of the classical 1D chain $P_\infty = P_{K-1} \approx P_{K-2}^2 \propto \mathcal{C}^N$ with the classical percolation transition point at $\mathcal{C}_t = 1$. The reason for the reduction is the decay of entanglement in the non-local entanglement links of higher hierarchy due to multiple entanglement swapping. The decay is exponential with respect to the hierarchy level $K - 1$, which implies polynomial decay of entanglement with respect to the chain length N , because $\alpha \leq 1/2$ and $N \propto 2^{K-1}$.

The effect of the polynomial decay of entanglement can be eliminated using standard protocol for entanglement distillation [25]. Reminding that any two-qubit state is distillable if entangled [26], let us estimate the amount of initial entanglement to implement the distillation in the hierarchical network. Let us consider the initial network configuration with K entangled pairs between the neighboring nodes and assuming that all qubit pairs (1) have the amount of entanglement of $\mathcal{C} = 1/2$, which corresponds to the states with Schmidt coefficients $\lambda_{1,2} = 1/2 \pm \sqrt{3}/4$. These states could be unitary transformed into Werner states [26] with fidelity of the each of the states $F \equiv \text{Tr}(|\varphi\rangle\langle\varphi||\Psi^+\rangle\langle\Psi^+|) = (1 + 2\sqrt{\lambda_1\lambda_2})/2 = 3/4$. A single entanglement swapping at an arbitrary node of the 1D chain results into one of the two states (4)-(5). While the state (5) requires no distillation, the fidelity of the state (4) is given by $F_{(0)} = 1/2 + 1/(4\sqrt{14}) \approx 0.57$. Using the iterative formula for the entanglement distillation

$$F_{(i+1)} = \frac{F_{(i)}^2 + \frac{1}{9}(1 - F_{(i)})^2}{F_{(i)}^2 + \frac{2}{3}F_{(i)}(1 - F_{(i)}) + \frac{5}{9}(1 - F_{(i)})^2}, \quad (9)$$

we find that $F_{(0)} \rightarrow F_{(8)} > F$, i.e. the entanglement distillation protocol allows to restore the fidelity to the level before the entanglement swapping in just eight iterations. Taking into account that the success probability of the distillation protocol approximates $1/4$, 32 entangled pairs are required to distill a single entanglement link after an entanglement swapping. In a network with $K - 1$ levels of hierarchy, the number of entangled pairs before distillation thus scales exponentially as $2^{K-2} \times 32^{K-1} \propto 2^{6K}$, but because $K \propto \log_2 N$, the total number of initial states to construct the hierarchical network with the average concurrence $\mathcal{C} \geq 1/2$ per entanglement link scales just polynomially $\propto N^6$. The distillation procedure is efficient, because targets achieving states with non-unit fidelity $F \geq 3/4$ at each entanglement link. If each entanglement link in the hierarchical quantum network has the

average concurrence $\mathcal{C} \geq 1/2$, the probability to establish a perfect entanglement path between two arbitrary distant nodes is strictly higher than zero as suggested by the percolation properties of the quantum network.

Suggested distillation procedure is not optimal and is shown to demonstrate practically appropriate (i.e. polynomial) scaling of the initial resources with the network size. More advanced distillation protocols [26] may lead to even better scaling. The optimal strategy for the entanglement distillation depends on the optimal strategy for the hierarchical network construction, which is unknown yet, as we mentioned early.

IV. CONCLUSION

We introduced quantum networks created with LOCC, which exhibit non-classical percolation properties and have quantum communication advantages over corresponding classical networks. Using the notion of quantum entanglement percolation we showed how to establish long-distance perfect entanglement between arbitrary two nodes in 1D chain with imperfect entangled pairs. Apart from the quantum communication benefits, we clearly demonstrated the distinction between the percolation properties of the physical network configuration (composed of nodes and channels) and the quantum network configuration (consisting of nodes and entanglement links). We showed that the percolation properties of these networks are dramatically different, although they both correspond to the physical 1D configuration. This result suggests a study of structural complexity of entanglement graphs that can be simulated on a given quantum network [27].

Presented approach of constructing the non-local hierarchical levels on simple underlying classical networks can be extended beyond 1D network configuration [17]. In particular, the 2D square lattice with the classical percolation transition point at $p = 1/2$ can be modified to a hierarchical network with the percolation transition point at $p = 5/32 \approx 0.16$. General analysis of the hierarchical networks is, however, challenging and requires development of new theoretical and numerical tools.

The hierarchical networks exhibit property of explosive percolation [24] – the sudden emergence of large-scale connectivity in a network [28]. The fact that the hierarchical networks may be created and operated locally opens intriguing possibilities for experimental testing of the explosive percolation.

Finally because the quantum networks exhibit new percolation transition points, we may expect all-new percolation properties, which may lead to the construction of new local theory of percolation in quantum networks with new unexpected technological applications.

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- [1] H.J. Kimble, *Nature* **453**, 1023 (2008).
 - [2] N. Gisin and R. Thew, *Nature Photon.* **1**, 165 (2007).
 - [3] H. Bhram, R. Cleve, J. Watrous and R. de Wolf, *Phys. Rev. Lett.* **87**, 167902 (2001).
 - [4] S. Perseguers, M. Lewenstein, A. Acin and J.I. Cirac, *Nature Phys.* **6**, 539 (2010).
 - [5] G. Manzano, F. Galve, G.L. Giorgi, E. Hernandez-Garcia and R. Zambrini, *Scientific Reports* **3**, 1439 (2013).
 - [6] T.D. Ladd, F. Jelezko, R. Laflamme, Y. Nakamura, C. Monroe and J.L. O'Brien, *Nature* **464**, 45 (2010).
 - [7] V. Giovannetti, S. Lloyd and L. Maccone, *Nature Photon.* **5**, 222 (2011).
 - [8] J.I. Cirac, P. Zoller, H.J. Kimble and H. Mabuchi, *Phys. Rev. Lett.* **78**, 3221 (1997).
 - [9] A. Acin, J.I. Cirac and M. Lewenstein, *Nature Phys.* **3**, 256 (2007).
 - [10] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, Ed. 2, 1994).
 - [11] K. Kieling and J. Eisert, *Lect. Notes Phys.* **762**, 287 (2009).
 - [12] S. Perseguers, J.I. Cirac, A. Acin, M. Lewenstein and J. Wehr, *Phys. Rev. A* **77**, 022308 (2008).
 - [13] M. Cuquet and J. Calsamiglia, *Phys. Rev. Lett.* **103**, 240503 (2009).
 - [14] S. Perseguers, D. Cavalcanti, G.J. Lapeyre Jr., M. Lewenstein and A. Acin, *Phys. Rev. A* **81**, 032327 (2010).
 - [15] M. Cuquet and J. Calsamiglia, *Phys. Rev. A* **83**, 032319 (2011).
 - [16] S. Boettcher, B. Goncalves and H. Guclu, *J. Phys. A: Math. Theor.* **41**, 252001 (2008).
 - [17] S. Boettcher, J.L. Cook and R.M. Ziff, *Phys. Rev. E* **80**, 041115 (2009).
 - [18] M.A. Nielsen and I.J. Chuang, *Quantum Computation and Quantum Information* (Cambridge Univ. Press, Cambridge, 2000).
 - [19] G. Vidal, *Phys. Rev. Lett.* **83**, 1046 (1999).
 - [20] W.K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998).
 - [21] J.-W. Pan, D. Bouwmeester, H. Weinfurter and A. Zeilinger, *Phys. Rev. Lett.* **80**, 3891 (1998).
 - [22] S. Bose, V. Vedral and P.I. Knight, *Phys. Rev. A* **60**, 194 (1999).
 - [23] D.J. Watts and S.H. Strogatz, *Nature* **393**, 440 (1998).
 - [24] S. Boettcher, V. Singh and R.M. Ziff, *Nature Commun.* **3**, 787 (2012).
 - [25] C.H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J.A. Smolin, and W.K. Wootters, *Phys. Rev. Lett.* **76**, 722 (1996).
 - [26] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
 - [27] M. Siomau, *AIP Conf. Proc.* **1742**, 030017 (2016).
 - [28] R.M. D'Souza and J. Nagler, *Nature Phys.* **11**, 531 (2015).